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On Analytical Solutions During Damped Wave Conduction and Relaxation in a Finite Slab Subject to the Convective Boundary Condition

Kal Renganathan Sharma

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Abstract This article describes the use of the final condition in the time domain to obtain bounded and physically reasonable solutions for the convective boundary condition for the case of a finite slab. The temperature overshoot problem is revisited for the convective boundary condition. The use of a physically realistic time condition is shown to remove the overshoot and lead to bounded solutions within Clausius's inequality. The ramifications of these findings are discussed. The method of separation of variables was used to obtain the analytical solution for the wave temperature. The governing equation for temperature, a hyperbolic partial differential equation (PDE) is multiplied by $\exp(\tau/2)$ that results in a hyperbolic PDE less the damping component. The wave temperature can be used to better understand the transient phenomena of heat conduction. For materials with large relaxation times, $\tau_{\mathrm{r}} > \frac{\rho C_p}{4h}$, the temperature can be expected to undergo subcritical damped oscillations. The analytical solution is presented as an infinite Fourier series solution. The solution was found to be bifurcated. For materials with a small relaxation time, the time domain part of the solution was found to be a decaying exponential and for materials with large relaxation times the time domain part of the solution was found to be cosinuous. Analytical solutions for the average temperature of the finite slab were also obtained. The thermal time constant of the material was found from the solution. The average temperature versus time was found to exhibit *convex curvature* for systems with large Biot numbers and the average temperature versus time was found to exhibit *concave curvature* for systems with small Biot numbers. The thermal time constant for the finite slab at different Biot numbers were found and tabulated. The thermal time constant versus Biot

K. R. Sharma (✉)
Department of Chemical Engineering, Prairie View A & M University,
PO Box 519, MS 2505, 77446 Prairie View, TX, USA
e-mail: jyoti_kalpika@yahoo.com; krsharma@pvamu.edu

number was found to exhibit a maxima. When Fourier parabolic equations are used, the thermal constant decreases monotonically with an increase in Biot number.

Keywords Convective boundary condition · Damped wave conduction and relaxation · Final condition · Heat waves · Hyperbolic PDE · Infinite Fourier series solution · Method of separation of variables · Subcritical damped oscillations · Taitel paradox · Wave temperature

List of Symbols

a	Half-width of a finite slab (m)
Bi	Biot number $(h\frac{\sqrt{\alpha\tau_r}}{k})$
$Bi*$	Biot number $(\frac{h}{Sa})$
C_p	Specific heat $(\mathrm{J\cdot kg^{-1}\cdot K^{-1}})$
h	Heat transfer coefficient between the fluid and the slab
k	Thermal conductivity $(\mathrm{W\cdot m^{-1}\cdot K^{-1}})$
q	Heat flux $(\mathrm{W\cdot m^{-2}})$
S	Storage coefficient $(\frac{\rho C_p}{\tau_r})$
t_c	Thermal time constant $(\tau_r\tau_c)$ [s]
T	Temperature (K)
T_1	Fluid temperature (K)
T_0	Initial temperature (K)
u	Dimensionless temperature $\frac{(T-T_1)}{(T_0-T_1)}$
$<u>$	Average temperature of the slab $\left(<u> = \frac{1}{X^*}\int_0^{X^*} u\mathrm{d}X\right)$ (K)
$V(\tau)$	Function of time only
V_h	Velocity of heat $\left(\sqrt{\frac{\alpha}{\tau_r}}\right)$
w	Wave temperature $(\mathrm{u} = \mathrm{w}\exp(-\tau/2))$
X	Dimensionless distance $\left(X = \frac{x}{\sqrt{\alpha\tau_r}}\right)$
$X*$	Dimensionless distance at $x = a$ $\left(X = \frac{a}{\sqrt{\alpha\tau_r}}\right)$

Greek Symbols

α	Thermal diffusivity $(\mathrm{m^2\cdot s^{-1}})$
τ_r	Relaxation time (s)
τ	Dimensionless time $(\frac{t}{\tau_r})$
τ_c	Dimensionless thermal time constant $(\frac{t_c}{\tau_r})$
ϕ	Function of space only

1 Introduction

Eight reasons were given to seek a generalized Fourier's law of heat conduction [1]. These include the contradiction of Fourier's law of heat conduction with the theory of microscopic reversibility of Onsager [2], oscillatory discharge of heat in good thermal conductors at low temperature [3], observation of Landau and Lifshitz [4] that the speed of heat cannot be greater than the speed of light, Casmir limit during nanoscale heat transfer [5], singularities found in mathematical models developed to describe transient events using Fourier's law of heat conduction [6], overprediction of theory compared to experiment found in a number of industrially important processes [7–11], and development of Fourier's law of heat conduction was based on empirical observations over a limited set of experimental conditions. The damped wave conduction and relaxation equation can be written as

$$q = -k\frac{\partial T}{\partial x} - \tau_{\mathrm{r}}\frac{\partial q}{\partial t} \tag{1}$$

When the relaxation time is zero, Eq. 1 will revert to Fourier's law of heat conduction. When combined with the energy balance equation in one dimension, Eq. 1 becomes in dimensionless form,

$$\frac{\partial u}{\partial \tau} + \frac{\partial^2 u}{\partial \tau^2} = \frac{\partial^2 u}{\partial X^2} \tag{2}$$

where $u = \frac{(T-T_1)}{(T_0-T_1)}$, $\tau = \frac{t}{\tau_{\mathrm{r}}}$, and $X = \frac{x}{\sqrt{\alpha\tau_{\mathrm{r}}}}$

When the rate of change of temperature with time is much greater than an exponential rise with time e^{τ}, the generalized Fourier's law of heat conduction equation will revert to the wave equation [12,13]. Reference to the generalized Fourier's law of heat conduction can be traced back to Maxwell [14], and Morse and Feshbach [15]. Cattaneo [16] and Vernotte [17] postulated this equation independently. This equation can be used to account for a finite speed of heat. Reviews on the generalized Fourier's law of heat conduction have been provided by Joseph and Preziosi [18], and Ozisik and Tzou [19]. Sharma [20–24] discussed the manifestation of the damped wave transport and relaxation equation in industrial applications and provided bounded analytical solutions within the constraints of the second law of thermodynamics.

It was shown that the generalized Fourier law of heat conduction can be derived by including the acceleration term in the free electron theory, the acceleration term in the Stokes–Einstein theory for molecular diffusion, by accounting for the accumulation term in the kinetic theory of gases and combining in series Hooke's elastic element and Newton's viscous element in viscoelastic theory. The relaxation time was found to be a third of the collision time of the electron. The velocity of heat was found to be identical with the velocity of mass derived from a kinetic representation of the pressure or the Maxwell representation of the speed of molecules. They derived a set of equations for length scales comparable to the mean free path of the molecule. Ali [25,26] used statistical mechanics and the kinetic theory and derived the generalized Fourier law of heat conduction for monatomic and diatomic gases. Glass and McRae [27] studied the variable specific heat and thermal relaxation parameter.

The relaxation time has been measured by Brown and Churchill [28], Peshkov [29], and Zehnder and Rosakis [30]. The relaxation mechanism is fundamental to thermal resonance that cannot be depicted by Fourier's law of heat conduction [31]. For a thermal wave speed around 900 m · s^{-1} in 4340 steel at 480 °C, the value of the relaxation time was found to be of the order of 10^{-11} s. A table for relaxation times for different materials at different temperatures and pressures are not available in the literature. Relaxation times for materials with a non-homogeneous inner structure were presented by Kaminski [32]. For sodium bicarbonate, they report a relaxation time of 29 s and 20 s for sand, and 54 s for ion exchange materials. Mitura et al. [33] claim that for the falling drying rate period the average time is of the order of several 1,000 s. For homogeneous substances, relaxation time values range from 10^{-8} s to 10^{-10} s for gases and 10^{-10} s to 10^{-12} s for liquids and dielectric solids as concluded by Sieniutycz [34]. Mitra et al. [35] presented experimental evidence of the wave nature of heat propagation in processed meat and demonstrated that the hyperbolic heat conduction model is an accurate representation on a macroscopic level of the heat conduction process in such a biological material. They report a relaxation time of the order of 16 s.

Some investigators have raised some concerns about violations of the second law of thermodynamics by the generalized Fourier law of heat conduction [36–39]. Taitel [37] attempted to obtain an analytical solution to the governing equation and found that the solution temperature for some values exceeded the boundary temperature indicating a possible violation of the Clausius inequality.

Al-Nimr et al. [40] discussed the "temperature overshoot" phenomena. Al-Nimr and Naji [41] applied the wave theory of heat conduction to explain the thermal behavior of anisotropic materials.

Al-Nimr et al. [42] used a perturbation technique for solution of the generalized equations governing the thermal behavior of thin metal films described by a hyperbolic two-step model. The generalized equations of the model contained diffusion terms in both the electron and lattice energy equations and assume that the incident laser radiation is absorbed by both the electron gas and solid lattice to account for the thermal behavior of semi-conducting and impure metals. The perturbation technique was utilized to eliminate the coupling between the electron and phonon energy equations when the normalized temperature differences between electrons and phonons is a small quantity which is true in materials exhibiting high coupling factors. Al-Nimr [43] discussed the non-equilibrium entropy production under the effect of a dual-phase-lag conduction model and the phase-lag effect [44]. They used the phase-lag concept in a study of the thermal behavior of lumped systems [45]. They looked at the effect of the hyperbolic heat conduction model on the thermal behavior of perfect and imperfect contact composite slabs under the effect of the hyperbolic heat conduction model. They have analyzed [46] the thermal behavior of a multilayer slab with imperfect contact using the dual-phase-lag heat conduction model. Ramadan and Al-Nimr [47,48] studied thermal wave reflection and transmission in a multilayer slab with imperfect contact using the dual-phase-lag model.

Haji-Sheik et al. [49] pointed out some anomalies in the hyperbolic heat equation. The transient instability, including the intrinsic transition from the desirable stability (neutral stability) to the ultimate unstable response was investigated by Tzou [50] for a wide spectrum of heating rates. Tzou confirmed the relaxation time results from the rate equation within the mainframe of the second law in non-equilibrium, irreversible thermodynamics. Schnaid [51] attempted to derive the governing equation for heat conduction with a finite speed of heat propagation directly from classical thermodynamics. Cai et al. [52] presented algebraically explicit analytical solutions of hyperbolic type heat conduction equations in three dimensions. Lin and Chen [53] sought numerical solutions of hyperbolic heat conduction in cylindrical and spherical systems. Antaki [54] examined the dual-phase-lag equation that was introduced by Tzou and provided an analytical solution for the case of a semi-infinite medium subject to constant wall flux boundary conditions. Lewandowsha and Malinowski [55] attempted to provide an analytical solution of the hyperbolic heat conduction equation for the case of a finite medium symmetrically heated on both sides using the method of Laplace transforms. Volz et al. [56] used a molecular dynamics numerical solution to test the validity of the generalized Fourier law of heat conduction. They confirmed the generalized law when considering heat flux fluctuations at equilibrium. Temperature overshoot and undershoot were discussed by Tan and Yang [57] during thermal propagation of thermal waves in a thin film under transient conditions. Tian [58] mentioned that the basic waveform of thermal waves is hyperbolic waves.

This article describes the use of the final condition in the time domain to obtain bounded and physically reasonable solutions for the convective boundary condition for the case of a finite slab. The temperature overshoot problem is revisited for the convective boundary condition. The use of a physically realistic time condition is shown to remove the overshoot and lead to bounded solutions within Clausius's inequality. The ramifications of these findings are discussed. The use of the final condition in time is evaluated to see whether this would lead to solutions within the constraints of the second law of thermodynamics. The analytical solution for the case of a semi-infinite medium subject to the convective boundary condition is developed. The average temperature in a finite slab subject to the convective boundary condition is studied at various Biot numbers. The thermal time constant for a finite slab is obtained.

2 Theory

2.1 Finite Slab With Convective Boundary Condition

Consider a finite slab (Fig. 1) at initial temperature T_0 subjected to sudden contact with a fluid at temperature T_1. The transient temperature as a function of space and time is obtained. The velocity of heat propagation is $V_h = \sqrt{\frac{\alpha}{\tau_r}}$. The initial and final conditions in time and boundary conditions in space are given by

$$t = 0, \quad -a \leq x \leq +a, \quad T = T_0 \tag{3}$$

$$t = \infty, \quad -a \leq x \leq +a, \quad T = T_1 \tag{4}$$

$$t > 0, \quad x = 0, \quad \frac{\partial T}{\partial x} = 0 \tag{5}$$

$$t > 0, \quad x = \pm a, \quad -k\frac{\partial T}{\partial x} = h\left(T_1 - T\right). \tag{6}$$

The energy balance on a thin spherical shell at x with a thickness Δx is developed. The governing equation can be obtained after eliminating q between the energy

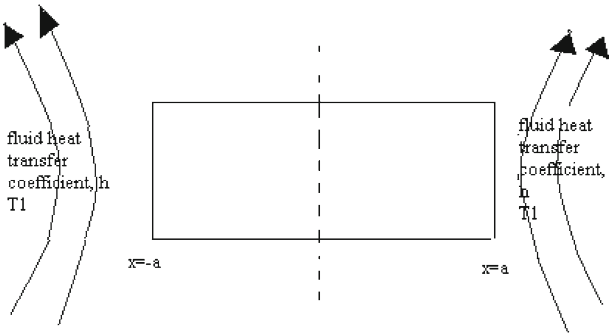

Fig. 1 Finite slab heated by fluid with heat transfer coefficient h

balance equation and the derivative with respect to x of the flux equation and introducing dimensionless variables;

$$\frac{\partial u}{\partial \tau} + \frac{\partial^2 u}{\partial \tau^2} = \frac{\partial^2 u}{\partial X^2}. \tag{7}$$

The solution is obtained by the method of separation of variables. First, Eq. 5 is multiplied by $\exp(n\tau)$;

$$\frac{\partial^2 (u\mathrm{e}^{n\tau})}{\partial X^2} = \mathrm{e}^{n\tau}\frac{\partial u}{\partial \tau} + \mathrm{e}^{n\tau}\frac{\partial^2 u}{\partial \tau^2}. \tag{8}$$

Let $w = u\mathrm{e}^{n\tau}$, then,

$$\begin{aligned} \frac{\partial w}{\partial \tau} &= \mathrm{e}^{n\tau}\frac{\partial u}{\partial \tau} + n\mathrm{e}^{n\tau}u = nw + \mathrm{e}^{n\tau}\frac{\partial u}{\partial \tau} \\ \frac{\partial^2 w}{\partial \tau^2} &= n\frac{\partial w}{\partial \tau} + n\mathrm{e}^{n\tau}\frac{\partial u}{\partial \tau} + \mathrm{e}^{n\tau}\frac{\partial^2 u}{\partial \tau^2}. \end{aligned} \tag{9}$$

Combining Eqs. 9 and 7,

$$\frac{\partial^2 w}{\partial X^2} = \frac{\partial w}{\partial \tau} - nw + \frac{\partial^2 w}{\partial \tau^2} - 2n\frac{\partial w}{\partial \tau} + n^2 w. \tag{10}$$

For $n = 1/2$, the damping term drops out and realizing that the wave temperature $w = \mathrm{u}\exp(\tau/2)$, Eq. 6 becomes

$$\frac{-w}{4} + \frac{\partial^2 w}{\partial \tau^2} = \frac{\partial^2 w}{\partial X^2}. \tag{11}$$

The method of separation of variables can be used to obtain the solution of Eq. 11.

$$\text{Let } \; W = V(\tau)\phi(X) \tag{12}$$

Inserting Eq. 12 into Eq. 11 and separating the variables that are a function of X and τ only,

$$\phi'' + \lambda^2 \phi = 0 \tag{13}$$

$$V'' = V\left(\frac{1}{4} - \lambda^2\right). \tag{14}$$

The solution for Eq. 13 can be written in a general form as

$$\phi = c_1 \sin(\lambda X) + c_2 \cos(\lambda X). \tag{15}$$

It can be seen that $c_1 = 0$ as the derivative of temperature with respect to X at $X = 0$ is 0.

Now from the boundary conditions (BC) at the surface,

$$\frac{\partial u}{\partial X} = -Biu \tag{16}$$

where

$$Bi = h\frac{\sqrt{\alpha \tau_{\mathrm{r}}}}{k} \tag{17}$$

$$-\lambda \sin(\lambda X) = -Bi \cos(\lambda X) \tag{18}$$

$$\frac{\lambda_n}{Bi} = \cot\left(\frac{\lambda_n a}{\sqrt{\alpha \tau_{\mathrm{r}}}}\right) \tag{19}$$

for small a,

$$\lambda_n = \sqrt{\frac{h\alpha\tau_{\mathrm{r}}}{ak}} + n\pi = \sqrt{\frac{h}{Sa}} + n\pi \tag{20}$$

where

$$\mathrm{S} = \frac{\rho C_p}{\tau_{\mathrm{r}}} \tag{21}$$

is the storage coefficient.

The time domain solution would be

$$V = \exp\left(-\frac{\tau}{2}\right)\left(c_3 \exp\left(\tau\sqrt{\frac{1}{4} - \lambda_n^2}\right) + c_4 \exp\left(-\tau\sqrt{\frac{1}{4} - \lambda_n^2}\right)\right) \tag{22}$$

$$or\ \ V \exp\left(\frac{\tau}{2}\right) = c_3 \exp\left(\tau\sqrt{\frac{1}{4} - \lambda_n^2}\right) + c_4 \exp\left(-\tau\sqrt{\frac{1}{4} - \lambda_n^2}\right). \tag{23}$$

From the final condition, $u = 0$ at infinite time. Thus, $V\phi \exp(\tau/2) = W$, the wave temperature at infinite time. The wave temperature is that portion of the solution that remains after dividing the damping component from either the solution or the governing equation. For any non-zero ϕ, it can be seen that at infinite time the LHS of Eq. 23 is a product of zero and infinity and a function of x and is zero. Hence, the RHS of Eq. 23 is also zero, and hence in Eq. 23, c_3 needs to be set to zero. Hence,

$$u = \sum_{1}^{\infty} c_n \exp\left(\frac{-\tau}{2}\right) \exp\left(-\tau\sqrt{\frac{1}{4} - \lambda_n^2}\right) \cos\left(\lambda_n X\right), \tag{24}$$

where λ_n is described by Eq. 20.

Now, when

$$\sqrt{\frac{h\alpha\tau_{\mathrm{r}}}{ak}} > 1/2 \tag{25}$$

$$\tau_{\mathrm{r}} > \frac{ak}{4h\alpha} \tag{26}$$

the pulsations in time domain will be seen. Then, the transient temperature profile will be given by

$$u = \sum_{1}^{\infty} c_n \exp\left(\frac{-\tau}{2}\right) \cos\left(\tau\sqrt{\lambda_n^2 - \frac{1}{4}}\right) \cos\left(\lambda_n X\right) \tag{27}$$

λ_n is given by Eq. 20. c_n can be obtained from the initial condition and the orthogonal property and is found to be $\frac{4(-1)^{n+1}}{(2n-1)\pi}$.

2.2 Average Temperature in a Finite Slab With Convective Boundary Condition

Consider a finite slab at a initial temperature of T_0 heated by air at a temperature of T_{air} and heat transfer coefficient, h. It is assumed that the heat transfer coefficient is constant with respect to time and the ambient temperature is held constant and at an elevated temperature compared with the initial temperature of the slab (Fig. 2). In

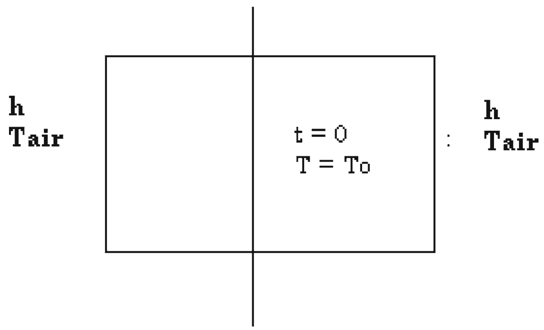

Fig. 2 Average temperature in a finite slab during convective heating

one dimension, the energy balance equation on a thin slice of thickness Δx can be written as

$$-\frac{\partial q}{\partial x} = \left(\rho C_p\right)\frac{\partial T}{\partial t}. \tag{28}$$

Combining Eq. 28 with the Cattaneo and Vernotte damped wave heat conduction and relaxation equation, the governing equation can be written as

$$\frac{\partial u}{\partial \tau} + \frac{\partial^2 u}{\partial \tau^2} = \frac{\partial^2 u}{\partial X^2}, \tag{29}$$

where

$$u = \frac{(T - T_{\text{air}})}{(T_0 - T_{\text{air}})}; \quad \tau = \frac{t}{\tau_{\text{r}}}; \quad X = \frac{x}{\sqrt{\alpha\tau_{\text{r}}}}. \tag{30}$$

The initial and final time conditions and space boundary conditions are

$$\tau = 0, \quad u = 1 \tag{31}$$

$$\tau = \infty, \quad u = 0 \tag{32}$$

$$X = 0, \quad \partial u/\partial X = 0 \tag{33}$$

$$X = \frac{a}{\sqrt{\alpha\tau_{\text{r}}}}$$

$$-h\left(T_{\text{air}} - < T >\right) = -k\frac{\partial T}{\partial x}. \tag{34}$$

The heat transfer coefficient is defined with respect to the average temperature in the slab for convenience of later illustration;

$$-Bi<u> = \frac{\partial u}{\partial X}, \tag{35}$$

where $Bi = \frac{h\sqrt{\alpha\tau_{\text{r}}}}{k} = \frac{\frac{\sqrt{\alpha\tau_{\text{r}}}}{k}}{\frac{1}{h}}$

The Biot number gives the ratio of the resistance to heat transfer by conduction corrected for thermal inertia effects to the resistance to heat transfer by convection. The convection is external to the slab, and conduction is internal to the slab. The conventional Biot number is modified with the penetration length $\sim(\alpha\tau_{\text{r}})$. The penetration length was obtained from a closed form solution to a step change in temperature at the surface to a semi-infinite medium by the method of relativistic transformation [13]. Equation 35 is integrated with respect to X between 0 and X*, where $X^* = a/\sqrt{(}\alpha\tau_{\text{r}})$ to give

$$\frac{\partial \langle u \rangle}{\partial \tau} + \frac{\partial^2 \langle u \rangle}{\partial \tau^2} = -\frac{Bi \langle u \rangle}{X^*} \tag{36}$$

$$\text{Let} \quad Bi^* = \frac{h\alpha\tau_r}{ak} = \frac{h}{Sa}, \tag{37}$$

where S = storage coefficient, $S = \frac{\rho C_p}{\tau_r}$. Equation 36 is a second-order ordinary differential equation (ODE) with constant coefficients. The solution to the ODE is

$$\langle u \rangle = \exp -\frac{\tau}{2}\left(a \exp\left(\frac{\tau}{2}\sqrt{1+4Bi^*}\right) + b \exp\left(-\frac{\tau}{2}\sqrt{1-4Bi^*}\right)\right) \tag{38}$$

or

$$\langle u \rangle \exp\frac{\tau}{2} = \left(a \exp\left(\frac{\tau}{2}\sqrt{1+4Bi^*}\right) + b \exp\left(-\frac{\tau}{2}\sqrt{1-4Bi^*}\right)\right) \tag{39}$$

From the final condition in time as given in Eq. 32, at infinite time, the LHS of Eq. 39 becomes zero multiplied by infinity and becomes zero. Hence, a can be seen to be zero. From the initial condition given by Eq. 31, the average initial temperature is also 1. Hence, $b = 1$. Thus, the average temperature in the finite slab is given by

$$\langle u \rangle = \exp\left(-\frac{\tau}{2}\right) \exp\left(-\frac{\tau}{2}\sqrt{1-4Bi^*}\right). \tag{40}$$

On examination of Eq. 40, when $Bi^* > 1/4$, the average temperature becomes subcritically damped and oscillatory. The argument within the square root becomes negative. The square root of -1 is i. Using De Movrie's theorem, $\exp(-i\theta) = \cos(\theta) - i \sin(\theta)$. The real part can be taken and for $Bi^* > 1/4 \frac{h\alpha\tau_r}{ak} > \frac{1}{4}$

$$\langle u \rangle = e^{-\frac{\tau}{2}} \cos\left(\frac{\tau\sqrt{4Bi^* - 1}}{2}\right). \tag{41}$$

The dimensionless frequency of the oscillations are $\sqrt{4Bi^* - 1}$. The frequency of the oscillations are $\frac{\sqrt{4Bi^*-1}}{\tau_r}$. As the Biot number is large, the frequency becomes large. As the relaxation time increases, the frequency becomes smaller. Equation 40 can be applied for $Bi < 1/4$ and Eq. 41 for $Bi > 1/4$. For $Bi^* = 1/4$, the average temperature decays exponentially and is given by $\exp(-\tau/2)$. The thermal time constant of the slab can be defined as the time taken to attain, say 2/3 of the final value. The thermal time constants for this definition for different Biot numbers are shown in Table 1. The thermal time constant for small Biot numbers are given by

$$\ln(\langle u \rangle) = -\left(\frac{\tau}{2}\right)\left(1 + \sqrt{1-4Bi^*}\right) \tag{42}$$

or

$$\tau_c = \frac{2\ln(3)}{1+\sqrt{1-4Bi^*}}, \tag{43}$$

Table 1 Thermal time constant for a finite slab with convective boundary condition

S. No.	Biot number	Thermal time constant (t/τ_r)
1.	0.01	1.1
2.	0.1	1.24
3.	0.25	2.2
4.	1	1.116
5.	2	0.795
6.	4	0.575

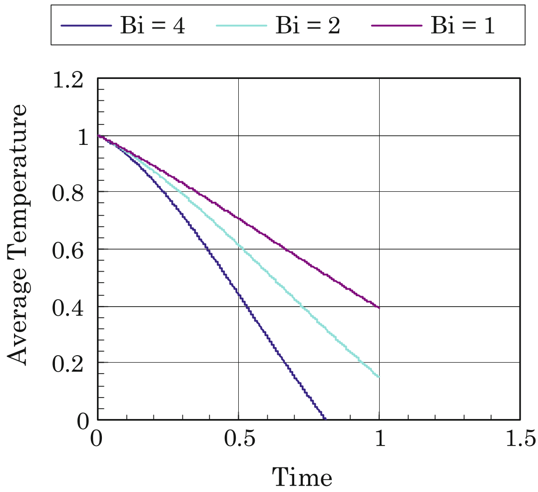

Fig. 3 Average temperature in a finite slab with convective boundary condition with $Bi > 1/4$

where τ_c is the thermal time constant as defined. As the Biot number is decreased, the thermal time constant decreases in this regime. This is for Biot numbers <0.25. Given the relaxation time, the time constant in s can be obtained by multiplying τ_c by the relaxation time. Thus, for a Biot number equal to 0.1 and a relaxation time equal to 15 s, the thermal time constant can be seen to be 18.6 s. The Biot number is given by the ratio of the heat transfer coefficient to the storage coefficient multiplied by the half-width of the slab. As the half-width is increased with the other parameters remaining the same, the Biot number decreases and the thermal time constant is found to decrease. For large Biot numbers as the half-width is increased, the thermal time constant increases.

It can be seen from the table that three different expressions were used to calculate the thermal time constant. The first regime is when the Biot number is small, this is for large slabs, a small heat transfer coefficient, and large storage coefficients. A large storage coefficient translates to small relaxation times and high thermal masses. The second regime is, when the Biot number is equal to 0.25 when the exponential decay in time, the solution for the transient temperature in dimensionless form, whereas the third regime is when the Biot number is greater than 0.25. This is for small slabs and a small storage coefficient. A small storage coefficient of the medium translates to large relaxation times and small thermal mass. In such cases, subcritical damped oscillations

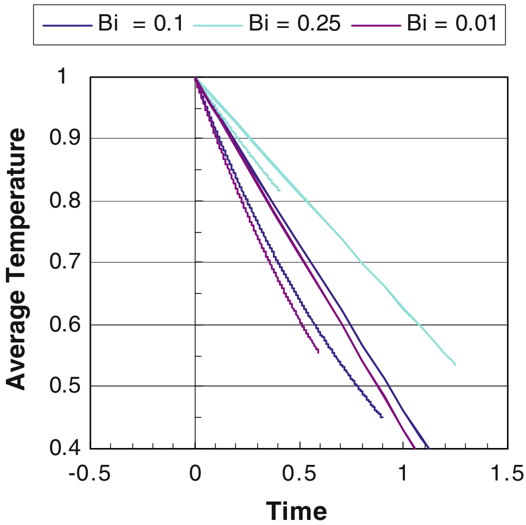

Fig. 4 Average temperature in a finite slab with convective boundary condition with $Bi \leq 1/4$

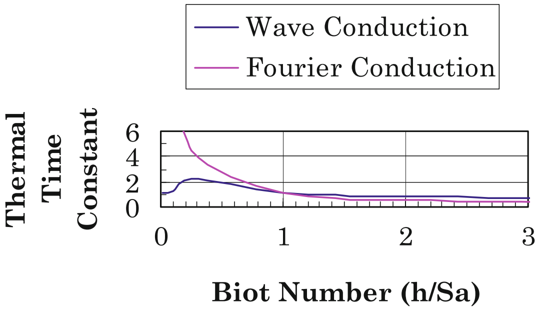

Fig. 5 Thermal time constant predictions as a function of Biot number (*h/Sa*)

can be found in the temperature in transience. Within the constraints of not having a negative temperature in the expression which would be a violation of the third law of thermodynamics, the time averaged temperature for a finite slab is shown for large and small Biot numbers in Figs. 3 and 4, respectively. There is a change from *convex curvature* to *concave curvature* from large Biot numbers to small Biot numbers. In the regime of small Biot numbers, the thermal time constant increases with increasing Biot number, whereas in the regime of large Biot numbers, the thermal time constant decreases with an increase in the Biot number. The thermal time constant as a function of the Biot number is shown in Fig. 5. The time taken to steady-state can be determined from the *x* intercepts in Figs. 3 and 4.

3 Conclusions

The physically reasonable, final condition in time is used, and the analytical solutions obtained were found to be within the bounds of the second law of thermodynamics. The method of separation of variables was used for the case of a finite slab subject to a convective boundary condition. It was shown that for materials with large relaxation times, the temperature will undergo subcritical damped oscillations. This is when the relaxation time, $\tau_r > \frac{a\rho C_p}{4h}$, where a is the half-width of the slab, (ρC_p) is the thermal mass, and h is the heat transfer coefficient of the fluid. The symmetric boundary condition was used in space. Suitable trigonometric approximations were used to obtain analytical expressions for the eigenvalues. The infinite Fourier series solution was found to be bifurcated. One solution with the decaying exponential in time is for materials with low relaxation times, and another solution with cosinuous time is for materials with large relaxation times. Analytical solutions for the average temperature of the finite slab were also obtained by the method of separation of variables. It was found that the slab reached a steady-state temperature after a finite elapsed time. From the x intercept of the graph for the average temperature versus time, the time taken to a steady-state was recorded. For systems with small Biot numbers, an analytical expression for the time taken to steady-state was derived. The thermal time constant of the materials which is the time taken for the slab to attain a steady-state temperature was tabulated as a function of different Biot numbers. The thermal time constant of the system versus the Biot number exhibited maxima as shown in Fig. 5. For systems with large Biot numbers, the average temperature versus time was found to exhibit *convex curvature*, and for systems with small Biot numbers, the average temperature versus time exhibit a *concave curvature*. Based on conditions that the average temperature is expected to undergo, subcritical damped oscillations were derived. These occur for materials with large relaxation times.

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